IDEAL-ADIC SEMI-CONTINUITY OF MINIMAL LOG DISCREPANCIES ON SURFACES

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ABSTRACT. We prove the ideal-adic semi-continuity of minimal log discrepancies on surfaces.

De Fernex, Ein and Mustaţă in [1] after Kollár in [4] proved the ideal-adic semi-continuity of log canonicity effectively, to obtain Shokurov's ACC conjecture [6] for log canonical thresholds on smooth varieties. Mustaţă formulated this semi-continuity for minimal log discrepancies.

Conjecture 1 (Mustaţă, see [3])**.** *Let (X,Δ) be a pair, Z a closed subset of X and $\mathscr{I}_Z$ its ideal sheaf. Let $\mathfrak{a}=\prod_{j=1}^{k}\mathfrak{a}_j^{r_j}$ be a formal product of ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. Then there exists an integer l such that the following holds: if $\mathfrak{b}=\prod_{j=1}^{k}\mathfrak{b}_j^{r_j}$ satisfies $\mathfrak{a}_j+\mathscr{I}_Z^l=\mathfrak{b}_j+\mathscr{I}_Z^l$ for all j, then*

$$\operatorname{mld}_Z(X,\Delta,\mathfrak{a})=\operatorname{mld}_Z(X,\Delta,\mathfrak{b}).$$

The case of minimal log discrepancy zero is the semi-continuity of log canonicity. Conjecture 1 is proved in the Kawamata log terminal (klt) case in [3, Theorem 1.6]. It is however inevitable to treat log canonical (lc) singularities in the study of limits of singularities; for example, the limit of klt pairs $(\mathbb{A}^2_{x,y},(x,y^n)\mathscr{O}_{\mathbb{A}^2})$ indexed by $n\in\mathbb{N}$ is the lc pair $(\mathbb{A}^2,x\mathscr{O}_{\mathbb{A}^2})$. The purpose of this paper is to settle Mustaţă's conjecture for surfaces.

Theorem 2. *Conjecture 1 holds when X is a surface.*

We must handle a non-klt triple $(X,\Delta,\mathfrak{a})$ which has positive minimal log discrepancy, but unlike the klt case, the log canonicity is no longer retained once when $\mathfrak{a}$ is expanded. However for surfaces, we are reduced to the purely log terminal (plt) case in which $\mathfrak{a}$ has an expression $\mathfrak{a}'\mathscr{O}_X(-C)$, then we can expand only the part $\mathfrak{a}'$ to apply the result on log canonicity.

We work over an algebraically closed field of characteristic zero. We use the notation below for singularities in the minimal model program.

Notation 3. A *pair* (X,Δ) consists of a normal variety X and an effective $\mathbb{R}$-divisor Δ such that $K_X+\Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We treat a *triple* $(X,\Delta,\mathfrak{a})$ by attaching a formal product $\mathfrak{a}=\prod_j\mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. A prime divisor E on a normal variety X' with a proper birational morphism $\varphi\colon X'\to X$ is called a divisor *over* X, and the image $\varphi(E)$ on X is called the *centre* of E on X and denoted by $c_X(E)$. We denote by $\mathscr{D}_X$ the set of divisors over X. The *log discrepancy* $a_E(X,\Delta,\mathfrak{a})$ of E is defined as $1+\operatorname{ord}_E(K_{X'}-\varphi^*(K_X+\Delta))-\operatorname{ord}_E\mathfrak{a}$. The triple $(X,\Delta,\mathfrak{a})$ is said to be *log canonical*, *Kawamata log terminal* if $a_E(X,\Delta,\mathfrak{a})\geq 0$, >0 respectively for all $E\in\mathscr{D}_X$, and said to be *purely log terminal*, *canonical*, *terminal* if $a_E(X,\Delta,\mathfrak{a})>0$, ≥ 1, >1 respectively for all exceptional $E\in\mathscr{D}_X$. A centre $c_X(E)$ with $a_E(X,\Delta,\mathfrak{a})\leq 0$ is called a *non-klt*

centre. Let Z be a closed subset of X. The *minimal log discrepancy* $\mathrm{mld}_Z(X,\Delta,\mathfrak{a})$ over Z is the infimum of $a_E(X,\Delta,\mathfrak{a})$ for all $E \in \mathscr{D}_X$ with centre in Z. We say that $E \in \mathscr{D}_X$ *computes* $\mathrm{mld}_Z(X,\Delta,\mathfrak{a})$ if $c_X(E) \subset Z$ and $a_E(X,\Delta,\mathfrak{a}) = \mathrm{mld}_Z(X,\Delta,\mathfrak{a})$ (or negative when $\mathrm{mld}_Z(X,\Delta,\mathfrak{a}) = -\infty$).

Prior to the proof of Theorem 2, we collect standard reductions and known results on Conjecture 1.

Lemma 4 ([3, Remarks 1.5.3, 1.5.4])**.** *Conjecture 1 is reduced to the case when X has $\mathbb{Q}$-factorial terminal singularities, $\Delta = 0$ and Z is irreducible*; *and it suffices to prove the inequality* $\mathrm{mld}_Z(X,\mathfrak{a}) \le \mathrm{mld}_Z(X,\mathfrak{b})$.

Theorem 5. *Conjecture 1 holds in each of the following cases.*

(i) $\mathrm{mld}_Z(X,\mathfrak{a}) = -\infty$.
(ii) (Kollár [4], de Fernex, Ein, Mustaţă [1]) $\mathrm{mld}_Z(X,\mathfrak{a}) = 0$.
(iii) ([3, Theorem 1.6]) $(X,\mathfrak{a})$ *is klt about Z.*

Remark 6. In (ii) above, one can take as l any integer greater than the maximum of $\mathrm{ord}_E\, \mathfrak{a}_j / \mathrm{ord}_E\, \mathscr{I}_Z$, by fixing $E \in \mathscr{D}_X$ which computes $\mathrm{mld}_Z(X,\mathfrak{a})$. The estimate of l in (iii) involves the log canonical threshold of $\mathfrak{a}$.

Conjecture 1 for surfaces is reduced to the plt case.

Lemma 7. *One may assume the following for Conjecture 1 for surfaces.*

(i) *X is a smooth surface, $\Delta = 0$ and Z is a closed point.*
(ii) *$(X,\mathfrak{a})$ is plt with unique non-klt centre C.*
(iii) *C is a smooth curve.*

Proof. We may assume that X is smooth with $\Delta = 0$ by Lemma 4, and may assume $\mathrm{mld}_Z(X,\mathfrak{a}) > 0$ by Theorem 5(i), (ii). Let C be the non-klt locus of $(X,\mathfrak{a})$. By Theorem 5(iii), we have only to work about $Z \cap C$. The assumption $\mathrm{mld}_Z(X,\mathfrak{a}) > 0$ means that Z contains no non-klt centre, whence $Z \cap C$ consists of finitely many closed points. By replacing Z with $Z \cap C$ and working locally, we may assume that Z is a closed point x, and $(X,\mathfrak{a})$ has the non-klt locus C which is a curve. The exceptional divisor E of the blow-up of X at x has positive log discrepancy $a_E(X,\mathfrak{a})$, but it is at most $a_E(X,C) = 2 - \mathrm{mult}_x C$. So C must be smooth at x. q.e.d.

We work locally about the closed point $x = Z$ with the assumptions in Lemma 7. We denote by $\mathfrak{m}$ the maximal ideal sheaf at x, and use the notation similar to [3, Definition 1.3].

Definition 8. For $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$ and $l \in \mathbb{N}$, we write $\mathfrak{a} \equiv_l \mathfrak{b}$ if $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$ for all j.

Set $c := \mathrm{mld}_x(X,\mathfrak{a})$. The non-trivial locus of $\mathfrak{a}$ is a divisor of form $C + D$. Since $(X,\mathfrak{a})$ is plt, we can fix $s,t > 0$ and $t' \ge 0$ such that $\mathrm{mld}_x(X, sD, \mathfrak{a}\mathfrak{m}^{t'}) = \mathrm{mld}_x(X,\mathfrak{a}\mathfrak{m}^t) = 0$. We fix a log resolution $\varphi\colon \bar{X} \to X$ of $(X,\mathfrak{a}\mathfrak{m})$, that is, $\prod_j \mathfrak{a}_j \mathfrak{m}\mathscr{O}_{\bar{X}}$ defines a divisor with simple normal crossing support. Let $\bar{C},\bar{D}$ denote the strict transform of C,D. Since C is smooth, $\bar{C}$ intersects only one prime divisor F in $\varphi^{-1}(x)$. This will play a crucial role in the proof. By blowing up $\bar{X}$ further, we may assume that every divisor E in $\varphi^{-1}(x)$ intersecting $\bar{D}$ satisfies

$$\mathrm{ord}_E\, D \ge s^{-1}c - 1. \tag{1}$$

We take an integer l such that

$$l > \mathrm{ord}_E \mathfrak{a}_j / \mathrm{ord}_E \mathfrak{m} \tag{2}$$

for all j and $E \subset \varphi^{-1}(x)$. The lemma below is an application of Theorem 5(ii) and Remark 6, with the inequality (2).

Lemma 9. $\mathrm{mld}_x(X, sD, \mathfrak{b}\mathfrak{m}^{t'}) = \mathrm{mld}_x(X, \mathfrak{b}\mathfrak{m}^t) = 0$ *for any* $\mathfrak{b} \equiv_l \mathfrak{a}$.

We write

$$\mathfrak{a}_j \mathscr{O}_{\bar{X}} = \mathscr{O}_{\bar{X}}(-H_j - V_j)$$

with divisors H_j, V_j such that $\mathrm{Supp}\, H_j \subset \bar{C} + \bar{D}$ and $\mathrm{Supp}\, V_j \subset \varphi^{-1}(x)$. Let $\mathfrak{b} \equiv_l \mathfrak{a}$. For $E \subset \varphi^{-1}(x)$, we have $\mathrm{ord}_E \mathfrak{a}_j < \mathrm{ord}_E \mathfrak{m}^l$ by (2), and $\mathrm{ord}_E \mathfrak{a}_j = \mathrm{ord}_E \mathfrak{b}_j$ by $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$. Hence we can write

$$\mathfrak{b}_j \mathscr{O}_{\bar{X}} = \mathfrak{b}'_j \mathscr{O}_{\bar{X}}(-V_j), \qquad \mathfrak{m}^l \mathscr{O}_{\bar{X}} = \mathscr{O}_{\bar{X}}(-M_j - V_j),$$

with an ideal sheaf $\mathfrak{b}'_j$ and an effective divisor M_j such that $\mathrm{Supp}\, M_j = \varphi^{-1}(x)$. Then the equality $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$ induces

$$\mathscr{O}_{\bar{X}}(-H_j) + \mathscr{O}_{\bar{X}}(-M_j) = \mathfrak{b}'_j + \mathscr{O}_{\bar{X}}(-M_j). \tag{3}$$

The following lemma shows $\mathrm{mld}_x(X, \mathfrak{b}) \geq c$, which with Lemma 4 completes Theorem 2.

Lemma 10. $a_G(X, \mathfrak{b}) \geq c$ *for any* $\mathfrak{b} \equiv_l \mathfrak{a}$ *and* $G \in \mathscr{D}_X$ *with* $c_X(G) = x$.

Proof. We divide into three cases according to the position of $c_{\bar{X}}(G)$.

(i) $c_{\bar{X}}(G) \not\subset \bar{C} + \bar{D}$.
(ii) $c_{\bar{X}}(G) \subset \bar{D}$.
(iii) $c_{\bar{X}}(G) \subset \bar{C}$.

(i) By (3), $\mathrm{Supp}\, H_j \cap \mathrm{Supp}\, M_j = \mathrm{Supp}\, \mathscr{O}_{\bar{X}}/\mathfrak{b}'_j \cap \mathrm{Supp}\, M_j$, whence $\mathrm{Supp}\, \mathscr{O}_{\bar{X}}/\mathfrak{b}'_j \cap \varphi^{-1}(x) \subset \bar{C} + \bar{D}$. In particular, $c_{\bar{X}}(G) \not\subset \mathrm{Supp}\, \mathscr{O}_{\bar{X}}/\mathfrak{b}'_j$. This implies $\mathrm{ord}_G \mathfrak{b}_j = \mathrm{ord}_G V_j = \mathrm{ord}_G \mathfrak{a}_j$, so $a_G(X, \mathfrak{b}) = a_G(X, \mathfrak{a}) \geq c$.

(ii) Take a prime divisor E in $\varphi^{-1}(x)$ such that $c_{\bar{X}}(G) \subset E$. By (1), $\mathrm{ord}_G D = \mathrm{ord}_E D \cdot \mathrm{ord}_G E + \mathrm{ord}_G \bar{D} \geq \mathrm{ord}_E D + 1 \geq s^{-1}c$. Lemma 9 for $(X, sD, \mathfrak{b}\mathfrak{m}^{t'})$ implies $a_G(X, \mathfrak{b}) \geq s\, \mathrm{ord}_G D$. These two inequalities induce $a_G(X, \mathfrak{b}) \geq c$.

(iii) $c_{\bar{X}}(G)$ is in the unique divisor $F \subset \varphi^{-1}(x)$ intersecting $\bar{C}$. There exists a divisor E in $\varphi^{-1}(x)$ with $a_E(X, \mathfrak{a}\mathfrak{m}^t) = 0$. Let L be the union of all such E. Then $L \cup \bar{C}$ is connected by the connectedness lemma [5, Theorem 17.4]. Hence $F \subset L$, that is, $a_F(X, \mathfrak{a}\mathfrak{m}^t) = 0$, so $\mathrm{ord}_F \mathfrak{m}^t = a_F(X, \mathfrak{a}) \geq c$ (actually $= c$ by precise inversion of adjunction [2]). Lemma 9 for $(X, \mathfrak{b}\mathfrak{m}^t)$ implies $a_G(X, \mathfrak{b}) \geq \mathrm{ord}_G \mathfrak{m}^t$. With $c_{\bar{X}}(G) \subset F$, we obtain $a_G(X, \mathfrak{b}) \geq \mathrm{ord}_G \mathfrak{m}^t \geq \mathrm{ord}_F \mathfrak{m}^t \geq c$. q.e.d.

Remark 11. The case division in the proof of Lemma 10 is in terms of the union H of divisors E with $\mathrm{ord}_E \mathfrak{a} > 0$ and $c_X(E) \not\subset Z$, on a suitable log resolution $\bar{X}$. We write $H = H' + H''$ so that H' is the union of those E with $a_E(X, \mathfrak{a}) = 0$. Then the cases (i), (ii), (iii) correspond to the conditions (i) $c_{\bar{X}}(G) \not\subset H$, (ii) $\subset H''$ and $\not\subset H'$, (iii) $\subset H'$ respectively. The proof of (i) works in any dimension, and (ii) works as long as $(X, \mathfrak{a})$ is plt (or more generally, dlt). However, (iii) would not work unless H' intersects only one divisor in $\varphi^{-1}(Z)$.

Remark 12. In [3], Conjecture 1 is formulated for $(X,\Delta,\mathfrak{a})$ with $\mathfrak{a}$ an $\mathbb{R}$-ideal sheaf as an equivalence class of formal products of ideal sheaves. Our proof is valid also for this formulation.

Acknowledgements. This paper was generated in the discussions during the workshop at American Institute of Mathematics. I am grateful to Professor T. de Fernex for his suggestion of the connectedness lemma after increasing the boundary. I thank Mr Y. Nakamura for his interest in the surface case and Professor M. Mustaţă for his conjecture. American Institute of Mathematics supported my participation financially. The research was partially supported by Grant-in-Aid for Young Scientists (A) 24684003.

REFERENCES

[1] T. de Fernex, L. Ein and M. Mustaţă, Shokurov's ACC conjecture for log canonical thresholds on smooth varieties, Duke Math. J. **152** (2010), 93-114

[2] L. Ein, M. Mustaţă and T. Yasuda, Jet schemes, log discrepancies and inversion of adjunction, Invent. Math. **153** (2003), 519-535

[3] M. Kawakita, Ideal-adic semi-continuity problem for minimal log discrepancies, arXiv:1012.0395

[4] J. Kollár, Which powers of holomorphic functions are integrable?, arXiv:0805.0756

[5] J. Kollár et al, *Flips and abundance for algebraic threefolds*, Astérisque **211** (1992)

[6] V. Shokurov, 3-fold log models, J. Math. Sci. **81** (1996), 2667-2699

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